2. gaiari buruzko autoebaluazioaren ebazpena

1. Kalkula itzazu honako limite hauek:

$$\text{(a)}\ \lim_{x\to 0}\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}};\quad \text{(b)}\ \lim_{x\to 0}\frac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)};\quad \text{(c)}\ \lim_{x\to\pi}\frac{1-\sin\frac{x}{2}}{\pi-x}.$$

Ebazpena:

(a)

$$\lim_{x\to 0}\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}}\overset{IB}{=}\lim_{x\to 0}\frac{1}{x}\cdot\frac{1}{2}\left(\frac{1+x}{1-x}-1\right)=\lim_{x\to 0}\frac{1}{2}\cdot\frac{2x}{x-x^2}=1.$$

(IB=infinitesimo baliokideak)

(b)

$$\lim_{x\to 0}\frac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)}\overset{L'H\hat{o}p}{=}\lim_{x\to 0}\frac{\dfrac{\dfrac{1}{(1-x^2)^{3/2}}}{\sqrt{1-\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}}}{\dfrac{-1}{1-x}}=\lim_{x\to 0}\frac{-1}{(1+x)\sqrt{1-2x^2}}=-1.$$ Goikoaren deribatua honela kalkulatzen da:

$$\left(\arcsin\frac{x}{\sqrt{1-x^2}}\right)'=\frac{1\cdot\sqrt{1-x^2}-x\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}\cdot\frac{1}{\sqrt{1-\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}}$$

(c)

$$\lim_{x\to\pi}\frac{1-\sin\frac{x}{2}}{\pi-x}\overset{L'H\hat{o}p}{=}\lim_{x\to\pi}\frac{-\frac{1}{2}\cos\frac{x}{2}}{-1}=0.\quad\square$$

2. Kalkula itzazu honako funtzio hauen deribatuak:

$$\text{(a)}\ y=e^{\ln\dfrac{x^2+3x}{2x+1}};\quad \text{(b)}\ y=\sin^{1/2}x\cos^3(x^2+1)\tan^{1/3}2x;\quad \text{(c)}\ y=\arctan(\ln\sqrt[3]{x^2}).$$

Ebazpena:

(a)

$$y'=\frac{2x^2+2x+3}{(x^2+3x)(2x+1)}e^{\ln\dfrac{x^2+3x}{2x+1}}=\frac{2x^2+2x+3}{(x^2+3x)(2x+1)}\cdot\frac{x^2+3x}{2x+1}=\frac{2x^2+2x+3}{(2x+1)^2}$$

(b)

$$y' = \left(\frac{1}{2}\frac{\cos x}{\sqrt{\sin x}}\right) \cdot (\cos^3(x^2+1)) \cdot (\tan^{1/3} 2x) +$$
$$(\sin^{1/2} x) \cdot \Big(6x[-\sin(x^2+1)]\cos^2(x^2+1)\Big) \cdot (\tan^{1/3} 2x) +$$
$$(\sin^{1/2} x) \cdot (\cos^3(x^2+1)) \cdot \left(\frac{2}{3\cos^2 2x}\tan^{-2/3} 2x\right).$$

(c)

$$y' = \frac{\frac{2}{3}x^{-1/3}}{x^{2/3}} \cdot \frac{1}{1+(\ln\sqrt[3]{x^2})^2} = \frac{2}{3x\Big(1+(\ln\sqrt[3]{x^2})^2\Big)}. \quad \square$$

3. Aurkitu $y'(x)$ honako funtzio inplizitu hauetarako:

(a) $y^4 - 3y^2 + x^2y - 4x^3 - 1 = 0$ (b) $\ln y + \sin y^2 + x^4 + 2 = 0$ (c) $\arctan y + y\sqrt{x^2+1} = 4$.

Ebazpena:

(a)

$$4y^3y' - 6yy' + 2xy + x^2y' - 12x^2 = 0 \iff y'(4y^3 - 6y + x^2) = -2xy + 12x^2$$
$$\iff y' = \frac{-2xy + 12x^2}{4y^3 - 6y + x^2}.$$

(b)

$$\frac{1}{y}y' + 2yy'\cos y^2 + 4x^3 = 0 \iff y'\left(\frac{1}{y} + 2y\cos y^2\right) = -4x^3$$
$$\iff y' = \frac{-4x^3}{\frac{1}{y} + 2y\cos y^2} = \frac{-4x^3y}{1 + 2y^2\cos y^2}.$$

(c)

$$\frac{1}{1+y^2}y' + y'\sqrt{x^2+1} + y\frac{2x}{2\sqrt{x^2+1}} = 0 \iff y'\left(\frac{1}{1+y^2} + \sqrt{x^2+1}\right) = -y\frac{2x}{2\sqrt{x^2+1}}$$
$$\iff y' = \frac{-y\frac{2x}{2\sqrt{x^2+1}}}{\frac{1}{1+y^2} + \sqrt{x^2+1}} = \frac{-yx}{\frac{\sqrt{x^2+1}}{1+y^2} + x^2 + 1}. \quad \square$$

4. Aurkitu a eta b-ren balioak honako funtzio hau jarraitua eta deribagarria izan dadin $\mathbb{R}$ multzoan:

$$f(x) = \begin{cases} ax^2 + bx - 1, & \text{baldin } x \leq 1 \\ 2bx - 2, & \text{baldin } x > 1. \end{cases}$$

Ebazpena: $f(x)$ jarraitua izateko $x=1$ puntuan honako hau bete behar da:

$$\left.\begin{aligned}\lim_{x\to 1^-} f(x) = a+b-1\\ \lim_{x\to 1^+} f(x) = 2b-2\end{aligned}\right\} \Rightarrow\ a+b-1=2b-2\ \Rightarrow\ b=a+1$$

$f(x)$ jarraitua denez $b=a+1$ denean eta hau dugunez:

$$f'(x) = \begin{cases} 2ax+b, & \text{baldin } x<1 \\ 2b, & \text{baldin } x>1, \end{cases}$$

$x=1$ puntuan $f(x)$ deribagarria izateko $f'(1^-)=f'(1^+)$ bete behar da, hau da:

$$\left.\begin{aligned}f'(1^-) = \lim_{x\to 1^-} f'(x) = 2a+(a+1)\\ f'(1^+) = \lim_{x\to 1^+} f'(x) = 2(a+1)\end{aligned}\right\} \Rightarrow\ 2a+(a+1)=2(a+1)\ \Rightarrow\ a=1\ \Rightarrow\ b=2.\ \square$$

5. Aurkitu $f(x)=10^x$-ren Taylor-en 3. ordenako garapena $a=0$ denean. Hori erabiliz, hurbil ezazu $10^{0.2}$. Zein da errorea? Nola egin dezakegu txikiago?

Ebazpena: $f(x)=10^x = 1+\dfrac{\ln 10}{1!}x+\dfrac{(\ln 10)^2}{2!}x^2+\dfrac{(\ln 10)^3}{3!}x^3+O(x^4)$

Beraz, $10^{0.2}=f(0.2)\approx 1+(\ln 10)0.2+\dfrac{1}{2}(\ln 10)^2(0.2)^2+\dfrac{1}{6}(\ln 10)^3(0.2)^3 = 1+0.4605170+0.1060380+0.0162774=1.5828324$

$10^{0.2}=1.5848932$ (kalkulagailu baten bidez).

Errorea$=|1.5848932-1.5828324|=0,0020608$

$n>3$ hartuz. $\square$

6. Kalkulatu honako integral hauek:

a) $\displaystyle\int \frac{\arctan\frac{x}{2}}{4+x^2}dx$; *b)* $\displaystyle\int \sin(\ln x)\frac{dx}{x}$; *c)* $\displaystyle\int \frac{x^2}{x^2+1}\arctan x dx$; *d)* $\displaystyle\int \frac{\ln x}{x^3}dx$;

e) $\displaystyle\int \frac{x^4dx}{x^4-1}$ *f)* $\displaystyle\int \frac{xdx}{x^2-7x+13}$; *g)* $\displaystyle\int \sin^5 x dx$; *h)* $\displaystyle\int \frac{\sin^3 x dx}{2+\cos x}$

Ebazpena:

a) Honako aldagai-aldaketa hau erabiliz:

$t=\arctan\dfrac{x}{2}\ \Rightarrow\ dt=2\dfrac{dx}{4+x^2}$

hau lortzen da:

$$\int \frac{\arctan\frac{x}{2}}{4+x^2}dx \overset{(t=\arctan\frac{x}{2})}{=} \int \frac{tdt}{2} = \frac{t^2}{4}+K \overset{(t=\arctan\frac{x}{2})}{=} \frac{(\arctan\frac{x}{2})^2}{4}+K.$$

b) Honako aldagai-aldaketa hau erabiliz:

$t = \ln x \;\Rightarrow\; dt = dt = \dfrac{dx}{x}$

hau lortzen da:

$$\int \sin(\ln x)\frac{dx}{x} \overset{(t=\ln x)}{=} \int \sin t dt = -\cos t + K \overset{(t=\ln x)}{=} -\cos(\ln x) + K$$

c)

$$I = \int \frac{x^2}{x^2+1}\arctan x dx = \int \left(1 - \frac{1}{1+x^2}\right)\arctan x dx = \int \arctan x dx - \int \frac{\arctan x}{1+x^2}dx$$

Lehenengo integralean zatikako integrazioa erabiliz, $u = \arctan x$ eta $dv = dx$ hartuz, honako hau lortzen da:

$$\int \arctan x dx = x\arctan x - \int \frac{xdx}{1+x^2} = x\arctan x - \frac{1}{2}\ln(1+x^2)$$

Bigarren integralean $t = \arctan x$ aldagai-aldaketa hartuz honako hau lortzen dugu:

$$\int \frac{\arctan x}{1+x^2}dx = \int t dt = \frac{1}{2}t^2 = \frac{1}{2}(\arctan x)^2.$$

Beraz, $I = x\arctan x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + K$.

d) Zatikako integrazioa erabiliz, $u = \ln x$ eta $dv = \dfrac{dx}{x^3}$ hartuz, honako hau lortzen da:

$$\int \frac{\ln x}{x^3}dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\int \frac{dx}{x^3} = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + K.$$

e) $$\int \frac{x^4 dx}{x^4-1} = x + \int \frac{dx}{x^4-1} = x + \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= x + \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + K.$$

f) $$\int \frac{xdx}{x^2-7x+13} = \frac{1}{2}\int \frac{(2x-7)+7}{x^2-7x+13}dx = \frac{1}{2}\ln(x^2-7x+13) + \frac{7}{2}\int \frac{dx}{(x-\frac{7}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{2}\ln(x^2-7x+13) + \frac{7}{\sqrt{3}}\arctan\frac{2(x-\frac{7}{2})}{\sqrt{3}} + K.$$

g) $$\int \sin^5 x dx = \int (1-\cos^2 x)^2 \sin x dx \overset{(t=\cos x)}{=} \int -(1-t^2)^2 dt = \int (-1+2t^2-t^4)dt$$

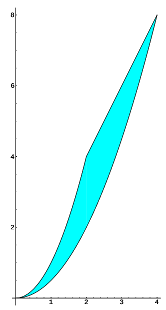
$$= -t + \frac{2t^3}{3} - \frac{t^5}{5} + K = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + K.$$

h) $\displaystyle\int \frac{\sin^3 x dx}{2+\cos x} = \int \frac{1-\cos^2 x}{2+\cos x}\sin x dx \overset{(t=\cos x)}{=} \int \frac{1-t^2}{2+t}(-dt) = \int (t-2)dt + \int \frac{3dt}{2+t} = \frac{t^2}{2} - 2t + 3\ln|2+t| + K = \frac{1}{2}\cos^2 x - 2\cos x + \ln|2+\cos x|^3 + K.$ □

7. Kalkulatu $y = x^2$, $y = x^2/2$ parabolak eta $y = 2x$ zuzenaren artean dagoen azalera.

Ebazpena: $x^2 = 2x \iff x(x-2) = 0 \iff x = 0$ edo $x = 2$.

$$x^2/2 = 2x \iff x(x-4) = 0 \iff x = 0 \quad \text{edo} \quad x = 4.$$



$$A_1 = \int_0^2 \left(x^2 - \frac{x^2}{2}\right) dx = \frac{4}{3}.$$
$$A_2 = \int_2^4 \left(2x - \frac{x^2}{2}\right) dx = \frac{8}{3}.$$
$$\text{Azalera} = A_1 + A_2 = \frac{4}{3} + \frac{8}{3} = 4.$$

8. Kalkulatu $y = e^x$, $x = 0$ eta $y = 0$-k mugaturiko gainazala OX-ren inguruan biratzean sortutako bolumena.

Ebazpena: $B = \pi \displaystyle\int_{-\infty}^0 e^{2x} dx = \pi \lim_{a\to-\infty} \int_a^0 e^{2x} dx = \pi \lim_{a\to-\infty} \frac{1}{2}[1 - e^{2a}] = \pi/2.$